Lecture 18

3.5 : Optimization problems

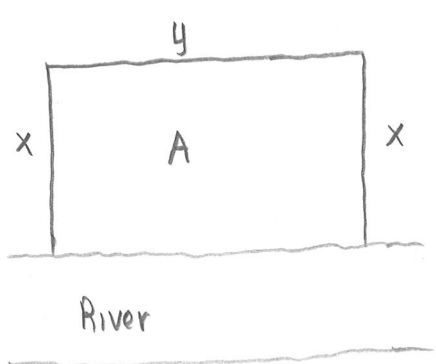
The methods we have learned in this chapter for finding extreme values have practical applications in many areas of life.

The biggest difficulty is converting word problems into mathematical problem i.e. setting up a function that needs to be maximized or minimized.

Read Pg 174 on Book

Ex A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Solution



We wish to maximize the area A of the rectangle

let x and y be the depth and width of a rectangle

Then, $A = xy$

We would like to express A as a function of just one variable,

we can express y as x and eliminate y.

To do so, we use the fact we have not used yet, that

the total length of the fencing is 2400 ft.

So, $2x + y = 2400$

$\Rightarrow y = 2400 - 2x$

Therefore, we have that $A(x) = (2400 - 2x)x = 2400x - 2x^2$

Note that $x \geq 0$ (because length has to be positive)

and $x \leq 1200$ (because otherwise y would be negative).

Therefore function we would like to maximize is

$A(x) = 2400x - 2x^2 \quad , \quad 0 \leq x \leq 1200$

(Since we have continuous function on a closed interval,

by extreme value thm, there exists a maximum value)

Then we need to find critical numbers.

$A'(x) = 2400 - 4x$, so to find the critical numbers we solve the equation $2400 - 4x = 0 \Rightarrow x = 600$.

Then the maximum value either occurs at this critical number or the endpoints.

$A(0) = 0$,

$A(600) = 600 \cdot (2400 - 2 \cdot 600) = 720,000$.

$A(1200) = 0$, the closed interval method gives the maximum value as $A(600) = 720,000$.

So the rectangular field should be 600 ft deep and 1200 ft wide. $\square$

We might not always have a closed interval to work with, so we would like to find a way to find absolute max/min of a function on an arbitrary interval.

First Derivative Test for absolute extreme values :

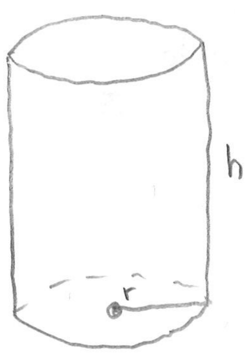
Spse that c is a critical number of a continuous function f defined on an interval.

a) If $f'(x) > 0$ for all $x < c$, and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the maximum value of f.

b) If $f'(x) < 0$ for all $x < c$, and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the minimum value of f.

Ex A cylindrical can is to be made to hold $1L$ of oil. Find the dimension that will minimize the cost of the metal to manufacture the can.

Soln

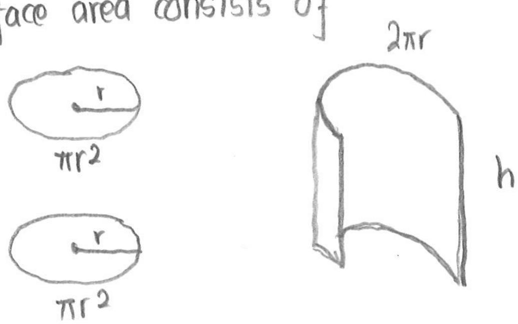


r = radius

h = height

In order to minimize the cost of the metal, we need to minimize the surface area of the cylinder (top, bottom, sides)

The surface area consists of



$2\pi rh$

So,

$$A = 2\pi r^2 + 2\pi rh$$

To eliminate h we use the fact that the volume is a given as $1\,L = 1000\ cm^3$

So $\pi r^2 h = 1000 \Rightarrow h = \dfrac{1000}{\pi r^2}$

$$A(r) = 2\pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2}$$

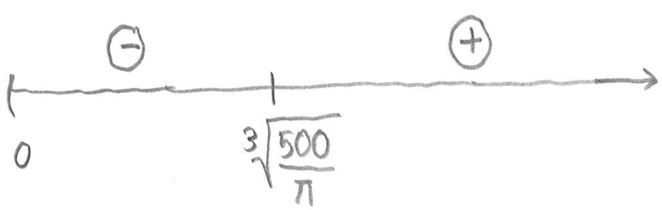
$$= 2\pi r^2 + \frac{2000}{r}, \quad r > 0.$$

To find the critical numbers, differentiate

$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}$$

The critical numbers are 0, and $\pi r^3 - 500 = 0 \Rightarrow r = \sqrt[3]{\dfrac{500}{\pi}}$

Since $r = 0$ is not in the domain of A, the only critical number is $\sqrt[3]{\dfrac{500}{\pi}}$



Hence $r = \sqrt[3]{\dfrac{500}{\pi}}$ minimizes surface area.

The corresponding value of h for $r = \sqrt[3]{\frac{500}{\pi}}$ is

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi}}\right)^2} = 2 \cdot \sqrt[3]{\frac{500}{\pi}} = 2r$$

☒

Alternate Solution (Using implicit differentiation)

$$A = 2\pi r^2 + 2\pi r h \quad , \quad \pi r^2 h = 1000$$

Insteadi of eliminating h, we differentiating both sides with respect to r.

$$A' = 4\pi r + 2\pi(h + r'h) \quad , \quad \pi(r^2 h' + 2rh) = 0$$

Set $A' = 0$

$$2r + h + r'h = 0 \qquad 2hr + r^2h' = 0$$

Since $r > 0$

$$2h + rh' = 0$$

$$2r + h - 2h = 0$$

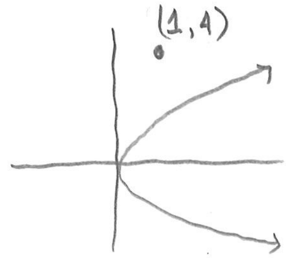
$$\boxed{h = 2r}$$

Ex 3

Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1,4)$

Soln The distance between point $(1,4)$ and point (x,y) is

$$d = \sqrt{(x-1)^2 + (y-4)^2}$$



But if the point (x,y) lies on the curve $y^2 = 2x$,

then $x = \frac{y^2}{2}$. So the point $(x,y) = \left(\frac{y^2}{2}, y\right)$

So the distance formula becomes

$$d = \sqrt{\left(\tfrac{1}{2}y^2 - 1\right)^2 + (y-4)^2}$$

Since distance is always positive, minimizing distance is the same as minimizing d^2.

$$d^2 = \left(\tfrac{1}{2}y^2 - 1\right)^2 + (y-4)^2 = f(y).$$

Then we need to find critical points.

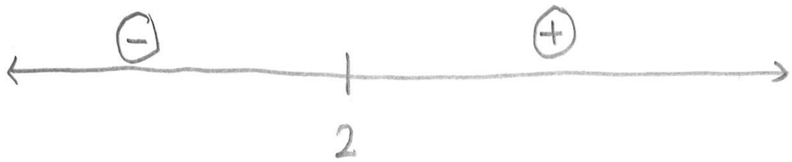
$$f'(y) = 2\left(\frac{1}{2}y^2 - 1\right) y + 2(y-4)$$

$$= (y^2 - 2) y + 2y - 8$$

$$= y^3 - 2y + 2y - 8$$

$$= y^3 - 8$$

So $f'(y) = 0 \Rightarrow y^3 - 8 = 0 \Rightarrow y = 2$



So minimum occurs when $y = 2$ and $x = \frac{2^2}{2} = 2$

So at pt $(2, 2)$.